

Infinite-dimensional Log-Determinant divergences between positive definite Hilbert-Schmidt operators

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Abstract

The current work generalizes the author's previous work on the infinite-dimensional Alpha Log-Determinant (Log-Det) divergences and Alpha-Beta Log-Det divergences, defined on the set of positive definite unitized trace class operators on a Hilbert space, to the entire Hilbert manifold of positive definite unitized Hilbert-Schmidt operators. This generalization is carried out via the introduction of the extended Hilbert-Carleman determinant for unitized Hilbert-Schmidt operators, in addition to the previously introduced extended Fredholm determinant for unitized trace class operators. The resulting parametrized family of Alpha-Beta Log-Det divergences is general and contains many divergences between positive definite unitized Hilbert-Schmidt operators as special cases, including the infinite-dimensional affine-invariant Riemannian distance and the infinite-dimensional generalization of the symmetric Stein divergence.

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infinite-dimensional Log-Determinant divergences, Alpha-Beta divergences, affine-invariant Riemannian distance, positive definite operators, Hilbert-Schmidt operators, extended Hilbert-Carleman determinant, trace class operators, extended Fredholm determinant

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1. Introduction

The current work is a continuation and generalization of the author's previous work [1], [2], which generalizes the finite-dimensional Log-Determinant divergences to the infinite-dimensional setting. We recall that for the convex cone $\text{Sym}^{++}(n)$ of symmetric, positive definite (SPD) matrices of size $n \times n$, $n \in \mathbb{N}$, the Alpha-Beta Log-Determinant (Log-Det) divergence between $A, B \in \text{Sym}^{++}(n)$ is a parametrized family of divergences defined by (see [3])

$$D^{(\alpha, \beta)}(A, B) = \frac{1}{\alpha\beta} \log \det \left[\frac{\alpha(AB^{-1})^\beta + \beta(AB^{-1})^{-\alpha}}{\alpha + \beta} \right], \alpha > 0, \beta > 0, \quad (1)$$

along with the limiting cases $(\alpha > 0, \beta = 0)$, $(\alpha = 0, \beta > 0)$, and $(\alpha = 0, \beta = 0)$. This family contains many distance-like functions on $\text{Sym}^{++}(n)$, including

1. The affine-invariant Riemannian distance d_{aiE} [4], corresponding to

$$D^{(0,0)}(A, B) = \frac{1}{2} d_{\text{aiE}}^2(A, B) = \frac{1}{2} \|\log(B^{-1/2}AB^{-1/2})\|_F^2, \quad (2)$$

where $\log(A)$ denotes the principal logarithm of the matrix A and $\|\cdot\|_F$ denotes the Frobenius norm. This is the geodesic distance associated with the so-called affine-invariant Riemannian metric [5, 6, 4, 7, 8].

2. The Alpha Log-Det divergences [9], corresponding to $D^{(\alpha, 1-\alpha)}(A, B)$, with

$$D^{(\alpha, 1-\alpha)}(A, B) = \frac{1}{\alpha(1-\alpha)} \log \left[\frac{\det[\alpha A + (1-\alpha)B]}{\det(A)^\alpha \det(B)^{1-\alpha}} \right], 0 < \alpha < 1, \quad (3)$$

$$D^{(1,0)}(A, B) = \text{tr}(A^{-1}B - I) - \log \det(A^{-1}B), \quad (4)$$

$$D^{(0,1)}(A, B) = \text{tr}(B^{-1}A - I) - \log \det(B^{-1}A). \quad (5)$$

The case $\alpha = 1/2$ gives the symmetric Stein divergence (also called the Jensen-Bregman LogDet divergence), whose square root is a metric on $\text{Sym}^{++}(n)$ [10], with $D^{(1/2, 1/2)}(A, B) = 4d_{\text{stein}}^2(A, B) = 4[\log \det(\frac{A+B}{2}) - \frac{1}{2} \log \det(AB)]$.

Previous work. In [1], we generalized the Alpha Log-Det divergences between SPD matrices [9] to the infinite-dimensional Alpha Log-Determinant divergences between positive definite unitized trace class operators on an infinite-dimensional Hilbert space. This is done via the introduction of the extended Fredholm determinant for

unitized trace class operators, along with the corresponding generalization of the log-concavity of the determinant for SPD matrices to the infinite-dimensional setting. In [2], we present a formulation for the Alpha-Beta Log-Det divergences between positive definite unitized trace class operators, generalizing the Alpha-Beta Log-Det divergences between SPD matrices as defined by Eq.(1). In both [1] and [2], for the divergences between reproducing kernel Hilbert spaces (RKHS) covariance operators, we obtain closed form formulas for the Alpha-Beta Log-Det divergences via the corresponding Gram matrices.

Contributions of this work. The current work is a continuation and generalization of [1] and [2]. In particular, we generalize the Alpha-Beta Log-Det divergences in [2] to the entire Hilbert manifold of positive definite unitized Hilbert-Schmidt operators on an infinite-dimensional Hilbert space. This is done by the introduction of the extended Hilbert-Carleman determinant for unitized Hilbert-Schmidt operators, in addition to the extended Fredholm determinant for unitized trace class operators employed in [1] and [2]. As in the finite-dimensional setting [3] and in [1], [2], the resulting family of divergences is general and admits as special cases many metrics and distance-like functions between positive definite unitized Hilbert-Schmidt operators, including the infinite-dimensional affine-invariant Riemannian distance in [11].

Comparison with the formulations in [1] and [2]. While the mathematical formulation presented in the current work, for Hilbert-Schmidt operators, is more general than the formulations in [1] and [2], which are for trace class operators, it should *not* be considered as a substitute for them. Many results in [1] and [2], especially those involving covariance operators, require explicitly the trace class assumption.

2. Positive definite unitized trace class and Hilbert-Schmidt operators

Throughout the paper, we assume that \mathcal{H} is a real separable Hilbert space, with $\dim(\mathcal{H}) = \infty$, unless explicitly stated otherwise. Let $\mathcal{L}(\mathcal{H})$ be the Banach space of bounded linear operators on \mathcal{H} , with operator norm $\| \cdot \|$. Let $\text{Sym}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ denote the subspace of bounded, self-adjoint operators on \mathcal{H} . Let $\text{Sym}^+(\mathcal{H}) \subset \text{Sym}(\mathcal{H})$ denote the set of self-adjoint, *positive* operators on \mathcal{H} , that is $A \in \text{Sym}^+(\mathcal{H}) \iff$

$\langle x, Ax \rangle \geq 0 \ \forall x \in \mathcal{H}$. Let $\text{Sym}^{++}(\mathcal{H}) \subset \text{Sym}^+(\mathcal{H})$ denote the set of self-adjoint, *strictly positive* operators on \mathcal{H} , that is $A \in \text{Sym}^{++}(\mathcal{H}) \iff \langle x, Ax \rangle > 0 \ \forall x \in \mathcal{H}, x \neq 0$, or equivalently, $\ker(A) = \{0\}$.

Most importantly, we consider the set $\mathbb{P}(\mathcal{H}) \subset \text{Sym}^{++}(\mathcal{H})$ of self-adjoint, bounded, *positive definite* operators on \mathcal{H} , which is defined by

$$A \in \mathbb{P}(\mathcal{H}) \iff A = A^*, \exists M_A > 0 \text{ such that } \langle x, Ax \rangle \geq M_A \|x\|^2 \ \forall x \in \mathcal{H}.$$

We use the notation $A > 0 \iff A \in \mathbb{P}(\mathcal{H})$.

In the following, let $\mathcal{C}_p(\mathcal{H})$ denote the set of p th Schatten class operators on \mathcal{H} (see e.g. [12]), under the norm $\|\cdot\|_p$, $1 \leq p \leq \infty$, which is defined by

$$\mathcal{C}_p(\mathcal{H}) = \{A \in \mathcal{L}(\mathcal{H}) : \|A\|_p = (\text{tr}|A|^p)^{1/p} < \infty\}, \quad (6)$$

where $|A| = (A^*A)^{1/2}$.

The cases we consider in this work are: (i) the space $\mathcal{C}_1(\mathcal{H})$ of trace class operators on \mathcal{H} , which we also denote by $\text{Tr}(\mathcal{H})$, and (ii) the space $\mathcal{C}_2(\mathcal{H})$ of Hilbert-Schmidt operators on \mathcal{H} , which we also denote by $\text{HS}(\mathcal{H})$.

Extended (unitized) trace class operators. In [1], we define the set of extended (or unitized) trace class operators on \mathcal{H} to be

$$\text{Tr}_X(\mathcal{H}) = \{A + \gamma I : A \in \text{Tr}(\mathcal{H}), \gamma \in \mathbb{R}\}.$$

The set $\text{Tr}_X(\mathcal{H})$ becomes a Banach algebra under the *extended trace class norm*

$$\|A + \gamma I\|_{\text{tr}_X} = \|A\|_{\text{tr}} + |\gamma| = \text{tr}|A| + |\gamma|.$$

For $(A + \gamma I) \in \text{Tr}_X(\mathcal{H})$, its *extended trace* is defined to be

$$\text{tr}_X(A + \gamma I) = \text{tr}(A) + \gamma.$$

By this definition $\text{tr}_X(I) = 1$, in contrast to standard trace definition, according to which $\text{tr}(I) = \infty$.

Extended (unitized) Hilbert-Schmidt operators. In [11], the author considered the following set of extended (unitized) Hilbert-Schmidt operators

$$\text{HS}_X(\mathcal{H}) = \{A + \gamma I : A \in \text{HS}(\mathcal{H}), \gamma \in \mathbb{R}\}. \quad (7)$$

The set $\text{HS}_X(\mathcal{H})$ can be equipped with the *extended Hilbert-Schmidt inner product* $\langle \cdot, \cdot \rangle_{\text{eHS}}$, defined by

$$\langle A + \gamma I, B + \mu I \rangle_{\text{eHS}} = \langle A, B \rangle_{\text{HS}} + \gamma\mu = \text{tr}(A^*B) + \gamma\mu.$$

along with the associated *extended Hilbert-Schmidt norm*

$$\|A + \gamma I\|_{\text{eHS}}^2 = \|A\|_{\text{HS}}^2 + \gamma^2 = \text{tr}(A^*A) + \gamma^2. \quad (8)$$

Under the inner product $\langle \cdot, \cdot \rangle_{\text{eHS}}$, the Hilbert-Schmidt operators are orthogonal to the scalar operators. Under the norm $\|\cdot\|_{\text{eHS}}$, $\|I\|_{\text{eHS}} = 1$, in contrast to the standard Hilbert-Schmidt norm, according to which $\|I\|_{\text{HS}} = \infty$.

Positive definite unitized trace class and Hilbert-Schmidt operators. The set of positive definite unitized trace class operators $\mathcal{P}\mathcal{C}_1(\mathcal{H}) \subset \text{Tr}_X(\mathcal{H})$ is defined to be the intersection

$$\mathcal{P}\mathcal{C}_1(\mathcal{H}) = \text{Tr}_X(\mathcal{H}) \cap \mathbb{P}(\mathcal{H}) = \{A + \gamma I > 0 : A^* = A, A \in \text{Tr}(\mathcal{H}), \gamma \in \mathbb{R}\}. \quad (9)$$

The set of positive definite unitized Hilbert-Schmidt operators $\mathcal{P}\mathcal{C}_2(\mathcal{H}) \subset \text{HS}_X(\mathcal{H})$ is defined to be the intersection

$$\mathcal{P}\mathcal{C}_2(\mathcal{H}) = \text{HS}_X(\mathcal{H}) \cap \mathbb{P}(\mathcal{H}) = \{A + \gamma I > 0 : A = A^*, A \in \text{HS}(\mathcal{H}), \gamma \in \mathbb{R}\}. \quad (10)$$

Remark 1. In [1] and [2], we use the notations $\text{PTr}(\mathcal{H})$ and $\Sigma(\mathcal{H})$ to denote $\mathcal{P}\mathcal{C}_1(\mathcal{H})$ and $\mathcal{P}\mathcal{C}_2(\mathcal{H})$, respectively. In the following, we refer to elements of $\mathcal{P}\mathcal{C}_1(\mathcal{H})$ and $\mathcal{P}\mathcal{C}_2(\mathcal{H})$ as *positive definite trace class operators* and *positive definite Hilbert-Schmidt operators*, respectively.

In [11], it is shown that the set $\mathcal{P}\mathcal{C}_2(\mathcal{H})$ assumes the structure of an infinite-dimensional Hilbert manifold and can be equipped with the following Riemannian metric. For each $P \in \mathcal{P}\mathcal{C}_2(\mathcal{H})$, on the tangent space $T_P(\mathcal{P}\mathcal{C}_2(\mathcal{H})) \cong \mathcal{H}_{\mathbb{R}} = \{A + \gamma I : A = A^*, A \in \text{HS}(\mathcal{H}), \gamma \in \mathbb{R}\}$, we define the following inner product

$$\langle A + \gamma I, B + \mu I \rangle_P = \langle P^{-1/2}(A + \gamma I)P^{-1/2}, P^{-1/2}(B + \mu I)P^{-1/2} \rangle_{\text{eHS}}.$$

The Riemannian metric given by $\langle \cdot, \cdot \rangle_P$ then makes $\mathcal{P}\mathcal{C}_2(\mathcal{H})$ an infinite-dimensional Riemannian manifold. Under this Riemannian metric, the geodesic distance between $(A + \gamma I), (B + \mu I) \in \mathcal{P}\mathcal{C}_2(\mathcal{H})$ is given by

$$d_{\text{aiHS}}[(A + \gamma I), (B + \mu I)] = \|\log[(B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}]\|_{\text{eHS}}. \quad (11)$$

Aim of this work. In [1], we introduce a parametrized family of divergences, called *Log-Determinant divergences*, between operators in $\mathcal{P}\mathcal{C}_1(\mathcal{H})$. In [2], we generalize these to the *Alpha-Beta Log-Determinant divergences* on $\mathcal{P}\mathcal{C}_1(\mathcal{H})$, which include the distance d_{aiHS} as a special case. However, these divergences are defined specifically on $\mathcal{P}\mathcal{C}_1(\mathcal{H})$. In the case $\dim(\mathcal{H}) = \infty$, the set $\mathcal{P}\mathcal{C}_1(\mathcal{H})$ of positive definite trace class operators on \mathcal{H} is a *strict subset* of the set of positive definite Hilbert-Schmidt operators $\mathcal{P}\mathcal{C}_2(\mathcal{H})$. In this work, we generalize the divergences in [1] and [2] to all of $\mathcal{P}\mathcal{C}_2(\mathcal{H})$.

3. Functions of positive definite unitized Hilbert-Schmidt operators

We first discuss several important functions on $\mathcal{P}\mathcal{C}_2(\mathcal{H})$, namely the exponential, logarithm, and power functions.

Exponential and logarithm functions. Consider the exponential function $\exp : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ defined by

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}. \quad (12)$$

In [11], it is shown that the map $\exp : \text{Sym}(\mathcal{H}) \cap \text{HS}_X(\mathcal{H}) \rightarrow \mathcal{P}\mathcal{C}_2(\mathcal{H})$ and its inverse function $\log = \exp^{-1} : \mathcal{P}\mathcal{C}_2(\mathcal{H}) \rightarrow \text{Sym}(\mathcal{H}) \cap \text{HS}_X(\mathcal{H})$ are diffeomorphisms. Here, for any $(A + \gamma I) \in \mathcal{P}\mathcal{C}_2(\mathcal{H})$, $\log(A + \gamma I)$ is defined via the spectral decomposition of A as follows. Let $\{\lambda_k\}_{k=1}^{\infty}$ be the eigenvalues of A with corresponding orthonormal eigenvectors $\{\phi_k\}_{k=1}^{\infty}$. Then

$$A = \sum_{k=1}^{\infty} \lambda_k \phi_k \otimes \phi_k, \quad \log(A + \gamma I) = \sum_{k=1}^{\infty} \log(\lambda_k + \gamma) \phi_k \otimes \phi_k, \quad (13)$$

where $\phi_k \otimes \phi_k : \mathcal{H} \rightarrow \mathcal{H}$ is a rank-one operator defined by $(\phi_k \otimes \phi_k)w = \langle \phi_k, w \rangle \phi_k$ $\forall w \in \mathcal{H}$. Since $\log(A + \gamma I) \in \text{Sym}(\mathcal{H}) \cap \text{HS}_X(\mathcal{H})$, it has the form

$$\log(A + \gamma I) = A_1 + \gamma_1 I, \quad A_1 \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H}), \gamma_1 \in \mathbb{R}.$$

Power functions. Given the exponential and logarithm functions, for any $\alpha \in \mathbb{R}$, the power function $(A + \gamma I)^\alpha$, for $(A + \gamma I) \in \mathcal{P}\mathcal{C}_2(\mathcal{H})$, is then well-defined via the following expression

$$(A + \gamma I)^\alpha = \exp[\alpha \log(A + \gamma I)] \in \mathcal{P}\mathcal{C}_2(\mathcal{H}).$$

Furthermore, for any two operators $(A + \gamma I), (B + \mu I) \in \mathcal{P}\mathcal{C}_2(\mathcal{H})$, we show that

$$\log[(A + \gamma I)(B + \mu I)^{-1}], \quad [(A + \gamma I)(B + \mu I)^{-1}]^\alpha, \alpha \in \mathbb{R} \quad (14)$$

are all well-defined and are elements of $\text{HS}_X(\mathcal{H})$ (though not necessarily of $\text{Sym}(\mathcal{H})$). To this end, let $B \in \mathcal{L}(\mathcal{H})$ be any invertible operator, then for any $A \in \mathcal{L}(\mathcal{H})$, we have

$$\exp(BAB^{-1}) = \sum_{j=0}^{\infty} \frac{(BAB^{-1})^j}{j!} = B \left(\sum_{j=0}^{\infty} \frac{A^j}{j!} \right) B^{-1} = B \exp(A) B^{-1}.$$

Thus for $(A + \gamma I) \in \mathcal{P}\mathcal{C}_2(\mathcal{H})$, the logarithm of $B(A + \gamma I)B^{-1} = BAB^{-1} + \gamma I \in \text{HS}_X(\mathcal{H})$ is also well-defined and is given by

$$\begin{aligned} \log[B(A + \gamma I)B^{-1}] &= B \log(A + \gamma I) B^{-1} \\ &= B(A_1 + \gamma_1 I)B^{-1} = BA_1B^{-1} + \gamma_1 I \in \text{HS}_X(\mathcal{H}). \end{aligned} \quad (15)$$

Using Eq. (15), we obtain the following results.

Proposition 1. *Let $(A + \gamma I), (B + \mu I) \in \mathcal{P}\mathcal{C}_2(\mathcal{H})$. Let $\Lambda + \frac{\gamma}{\mu} I = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}$. Then*

1. *The logarithm function $\log[(A + \gamma I)(B + \mu I)^{-1}] \in \text{HS}_X(\mathcal{H})$ is well-defined and is given by*

$$\log[(A + \gamma I)(B + \mu I)^{-1}] = (B + \mu I)^{1/2} \log \left(\Lambda + \frac{\gamma}{\mu} I \right) (B + \mu I)^{-1/2}. \quad (16)$$

2. For any $\alpha \in \mathbb{R}$, the power function $[(A + \gamma I)(B + \mu I)^{-1}]^\alpha \in \text{HS}_X(\mathcal{H})$ is well-defined and is given by

$$[(A + \gamma I)(B + \mu I)^{-1}]^\alpha = (B + \mu I)^{1/2} \left(\Lambda + \frac{\gamma}{\mu} I \right)^\alpha (B + \mu I)^{-1/2}. \quad (17)$$

4. The extended Hilbert-Carleman determinant

The key concept for defining Log-Determinant divergences between operators is determinant. We recall that for $A \in \text{Tr}(\mathcal{H})$, the Fredholm determinant $\det(I + A)$ is (see e.g. [13])

$$\det(I + A) = \prod_{k=1}^{\infty} (1 + \lambda_k), \quad (18)$$

where $\{\lambda_k\}_{k=1}^{\infty}$ are the eigenvalues of A . To define Log-Determinant divergences between positive definite trace class operators in $\mathcal{PC}_1(\mathcal{H})$, in [1], we generalize the Fredholm determinant to the *extended Fredholm determinant* of extended trace class operators. For $(A + \gamma I) \in \text{Tr}_X(\mathcal{H})$, $\gamma \neq 0$, its extended Fredholm determinant is defined to be, assuming that $\dim(\mathcal{H}) = \infty$,

$$\det_X(A + \gamma I) = \frac{1}{\gamma} \det \left(\frac{A}{\gamma} + I \right),$$

where the determinant on the right hand side is the Fredholm determinant (we refer to [1] for the derivation leading to this definition). For $\gamma = 1$, we recover the Fredholm determinant. In the case $\dim(\mathcal{H}) < \infty$, we define $\det_X(A + \gamma I) = \det(A + \gamma I)$, the standard matrix determinant.

The extended Fredholm determinant continues to play a key role in the current work, but it is not sufficient for dealing with positive definite Hilbert-Schmidt operators in $\mathcal{PC}_2(\mathcal{H})$. In order to do so, we introduce the concept of *extended Hilbert-Carleman determinant*.

We first recall the concept of the Hilbert-Carleman determinant for operators of the form $I + A$, where A is a Hilbert-Schmidt operator (see e.g. [13] for a comprehensive treatment). Following [13], for any bounded operator $A \in \mathcal{L}(\mathcal{H})$, consider the operator

$$R_n(A) = \left[(I + A) \exp \left(\sum_{k=1}^{n-1} \frac{(-A)^k}{k} \right) \right] - I. \quad (19)$$

If $A \in \mathcal{C}_n(\mathcal{H})$, then $R_n(A) \in \mathcal{C}_1(\mathcal{H})$. Thus the following quantity is well-defined

$$\det_n(I + A) = \det(I + R_n(A)). \quad (20)$$

In particular, for $n = 1$, we obtain $R_1(A) = A$ and thus

$$\det_1(I + A) = \det(I + A). \quad (21)$$

For $n = 2$, we have $R_2(A) = (I + A) \exp(-A) - I$ and thus

$$\det_2(I + A) = \det[(I + A) \exp(-A)]. \quad (22)$$

This is called the *Hilbert-Carleman determinant* of $I + A$. In particular, for $A \in \text{Tr}(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})$, we have

$$\det_2(I + A) = \det(I + A) \exp(-\text{tr}(A)), \quad (23)$$

$$\log \det_2(I + A) = \log \det(I + A) - \text{tr}(A). \quad (24)$$

The function $\det_2(I + A)$ is continuous in the Hilbert-Schmidt norm, so that

$$\lim_{k \rightarrow \infty} \|A_k - A\|_{\text{HS}} = 0 \Rightarrow \lim_{k \rightarrow \infty} \det_2(I + A_k) = \det_2(I + A). \quad (25)$$

We first have the following result.

Lemma 1. *Assume that $A \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ such that $I + A > 0$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be the eigenvalues of A . Then*

$$\log \det_2(I + A) = \sum_{k=1}^{\infty} [\log(1 + \lambda_k) - \lambda_k] \quad (26)$$

is well-defined and finite. Furthermore,

$$\log \det_2(I + A) \leq 0, \quad (27)$$

with equality if and only if $A = 0$.

The Hilbert-Carleman determinant \det_2 is defined for operators of the form $A + I$, $A \in \text{HS}(\mathcal{H})$, but not for operators of the form $A + \gamma I$, $\gamma > 0$, $\gamma \neq 1$. In the following, we generalize \det_2 to handle these operators. We first have the following generalization of the function $R_2(A) = (I + A) \exp(-A) - I$ above.

Lemma 2. Assume that $(A + \gamma I) \in \text{HS}_X(\mathcal{H})$, $\gamma \neq 0$. Define

$$R_{2,\gamma}(A) = (A + \gamma I) \exp(-A/\gamma) - \gamma I. \quad (28)$$

Then $R_{2,\gamma}(A) \in \text{Tr}(\mathcal{H})$ and hence $R_{2,\gamma}(A) + \gamma I = (A + \gamma I) \exp(-A/\gamma) \in \text{Tr}_X(\mathcal{H})$.

This also implies that the infinite product

$$\prod_{k=1}^{\infty} [(\lambda_k + \gamma) \exp(-\lambda_k/\gamma) - \gamma + 1] \quad (29)$$

converges to a finite value, where $\{\lambda_k\}_{k=1}^{\infty}$ are the eigenvalues of A .

In particular, for $\gamma = 1$, we have $R_{2,1}(A) = R_2(A)$. Motivated by Lemma 2 and the definition of \det_2 , we arrive at the following generalization of \det_2 .

Definition 1 (Extended Hilbert-Carleman determinant). For $(A + \gamma I) \in \text{HS}_X(\mathcal{H})$, $\gamma \neq 0$, its extended Hilbert-Carleman determinant is defined to be

$$\det_{2X}(A + \gamma I) = \det_X[R_{2,\gamma}(A) + \gamma I] = \det_X[(A + \gamma I) \exp(-A/\gamma)]. \quad (30)$$

If $\gamma = 1$, then we recover the Hilbert-Carleman determinant

$$\det_{2X}(A + I) = \det[(A + I) \exp(-A)] = \det_2(A + I). \quad (31)$$

If $(A + \gamma I) \in \text{Tr}_X(\mathcal{H})$, then

$$\det_{2X}(A + \gamma I) = \det_X(A + \gamma I) \exp(-\text{tr}(A)/\gamma). \quad (32)$$

The following are the some of the properties of \det_{2X} which we employ later on.

Lemma 3 (Factorization Rule).

$$\det_{2X}(A + \gamma I) = \gamma \det_2\left(\frac{A}{\gamma} + I\right). \quad (33)$$

If $(A + \gamma I) \in \text{Tr}_X(\mathcal{H})$, $\gamma \neq 0$, then Lemma 5 in [1] states that for any invertible operator $C \in \mathcal{L}(\mathcal{H})$, we have

$$\det_X[C(A + \gamma I)C^{-1}] = \det_X(A + \gamma I). \quad (34)$$

This property generalizes for \det_{2X} , with $(A + \gamma I) \in \text{HS}_X(\mathcal{H})$, as follows.

Lemma 4 (Similarity Invariant). *Let $(A + \gamma I) \in \text{HS}_X(\mathcal{H})$, $\gamma \neq 0$. Let $C \in \mathcal{L}(\mathcal{H})$ be invertible. Then*

$$\det_{2X}[C(A + \gamma I)C^{-1}] = \det_{2X}(A + \gamma I). \quad (35)$$

For $(A + \gamma I), (B + \mu I) \in \text{Tr}_X(\mathcal{H})$, we show in Proposition 4 in [1] that the product rule for determinants holds, that is $\det_X[(A + \gamma I)(B + \mu I)] = \det_X(A + \gamma I)\det_X(B + \mu I)$. For \det_2 and \det_{2X} and $(A + \gamma I), (B + \mu I) \in \text{HS}_X(\mathcal{H})$, this is no longer true in general. However, if $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_2(\mathcal{H})$, or if $(A + \gamma I), (B + \mu I) \in \text{Tr}_X(\mathcal{H})$, then we still have commutativity, that is $\det_{2X}[(A + \gamma I)(B + \mu I)] = \det_{2X}[(B + \mu I)(A + \gamma I)]$, as follows.

Lemma 5 (Commutativity). *Assume that $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_2(\mathcal{H})$. Then*

$$\det_{2X}[(A + \gamma I)(B + \mu I)] = \det_{2X}[(A + \gamma I)^{1/2}(B + \mu I)(A + \gamma I)^{1/2}] \quad (36)$$

$$= \det_{2X}[(B + \mu I)(A + \gamma I)] \quad (37)$$

$$= \det_{2X}[(B + \mu I)^{1/2}(A + \gamma I)(B + \mu I)^{1/2}]. \quad (38)$$

If $(A + \gamma I), (B + \mu I) \in \text{Tr}_X(\mathcal{H})$, $\gamma \neq 0, \mu \neq 0$, then

$$\det_{2X}[(A + \gamma I)(B + \mu I)] = \det_{2X}[(B + \mu I)(A + \gamma I)]. \quad (39)$$

An immediate consequence of Lemma 5 is the following.

Corollary 1 (Cyclic Property). *Assume that $(A + \gamma I), (B + \mu I), (C + \nu I) \in \mathcal{PC}_2(\mathcal{H})$, or $(A + \gamma I), (B + \mu I), (C + \nu I) \in \text{Tr}_X(\mathcal{H})$. Then*

$$\det_{2X}[(A + \gamma I)(B + \mu I)(C + \nu I)] = \det_{2X}[(C + \nu I)(A + \gamma I)(B + \mu I)] \quad (40)$$

$$= \det_{2X}[(B + \mu I)(C + \nu I)(A + \gamma I)]. \quad (41)$$

For the following properties, we assume explicitly that $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_2(\mathcal{H})$, that is $(A + \gamma I) > 0, (B + \mu I) > 0$ and $A, B \in \text{HS}(\mathcal{H})$. These properties are utilized in the formulation of the Log-Determinant divergences in Section 5.

Lemma 6. Let $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_2(\mathcal{H})$. Let $\Lambda + \nu I = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}$, where $\Lambda \in \text{HS}(\mathcal{H})$ and $\nu = \frac{\gamma}{\mu}$. Then for any $\alpha \in \mathbb{R}$,

$$\begin{aligned} \det_{2X}[(A + \gamma I)(B + \mu I)^{-1}]^\alpha &= \det_{2X}[(\Lambda + \nu I)^\alpha] \\ &= \det_{2X}[(B + \mu I)^{-1}(A + \gamma I)^\alpha]. \end{aligned} \quad (42)$$

Lemma 7. Let $(I + A) \in \mathcal{PC}_2(\mathcal{H})$. Let $\alpha \in \mathbb{R}$ be arbitrary. Then $(I + A)^\alpha - I \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$. Let $\{\lambda_k\}_{k=1}^\infty$ be the eigenvalues of A . Then the following quantities converge to finite values:

$$\det_2[(I + A)^\alpha] = \prod_{k=1}^\infty (1 + \lambda_k)^\alpha \exp[1 - (1 + \lambda_k)^\alpha], \quad (43)$$

$$\log \det_2[(I + A)^\alpha] = \sum_{k=1}^\infty [\alpha \log(1 + \lambda_k) + 1 - (1 + \lambda_k)^\alpha]. \quad (44)$$

Furthermore,

$$\log \det_2[(I + A)^\alpha] \leq 0, \quad (45)$$

with equality if and only if $A = 0$. For $(I + A) \in \mathcal{PC}_1(\mathcal{H})$,

$$\det_2[(I + A)^\alpha] = \det[(I + A)^\alpha] \exp(-\text{tr}[(I + A)^\alpha - I]), \quad (46)$$

$$\log \det_2[(I + A)^\alpha] = \alpha \log \det(I + A) - \text{tr}[(I + A)^\alpha - I]. \quad (47)$$

Lemma 8. Let $(A + \gamma I) \in \mathcal{PC}_2(\mathcal{H})$. Let $\alpha \in \mathbb{R}$ be arbitrary. Then $(A + \gamma I)^\alpha - \gamma^\alpha I \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$. Let $\{\lambda_k\}_{k=1}^\infty$ be the eigenvalues of A . Then the following quantities converge to finite values:

$$\det_{2X}[(A + \gamma I)^\alpha] = \gamma^\alpha \prod_{k=1}^\infty \left(1 + \frac{\lambda_k}{\gamma}\right)^\alpha \exp \left[1 - \left(1 + \frac{\lambda_k}{\gamma}\right)^\alpha\right], \quad (48)$$

$$\log \det_{2X}[(A + \gamma I)^\alpha] = \alpha \log \gamma + \sum_{k=1}^\infty \left[\alpha \log \left(1 + \frac{\lambda_k}{\gamma}\right) + 1 - \left(1 + \frac{\lambda_k}{\gamma}\right)^\alpha \right]. \quad (49)$$

For $(A + \gamma I) \in \mathcal{PC}_1(\mathcal{H})$,

$$\det_{2X}[(A + \gamma I)^\alpha] = \gamma^\alpha \det \left[\left(\frac{A}{\gamma} + I \right)^\alpha \right] \exp \left(-\text{tr} \left[\left(\frac{A}{\gamma} + I \right)^\alpha - I \right] \right), \quad (50)$$

$$\begin{aligned} \log \det_{2X}[(A + \gamma I)^\alpha] &= \alpha \log \gamma + \alpha \log \det \left(\frac{A}{\gamma} + I \right) - \text{tr} \left[\left(\frac{A}{\gamma} + I \right)^\alpha - I \right] \\ &= \alpha \log \det_X(A + \gamma I) - \text{tr} \left[\left(\frac{A}{\gamma} + I \right)^\alpha - I \right]. \end{aligned} \quad (51)$$

5. Infinite-dimensional Log-Determinant divergences between positive definite Hilbert-Schmidt operators

In [2], we define the Log-Determinant divergences between two positive definite trace class operators $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_1(\mathcal{H})$ as follows

$$\begin{aligned} D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] \\ = \frac{1}{\alpha \beta} \log \left[\left(\frac{\gamma}{\mu} \right)^{r(\delta - \frac{\alpha}{\alpha + \beta})} \det_X \left(\frac{\alpha(\Lambda + \frac{\gamma}{\mu} I)^{r(1-\delta)} + \beta(\Lambda + \frac{\gamma}{\mu} I)^{-r\delta}}{\alpha + \beta} \right) \right]. \end{aligned} \quad (52)$$

Here $\alpha > 0, \beta > 0, r \neq 0$ are fixed, $\Lambda + \frac{\gamma}{\mu} I = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}$, and $\delta = \frac{\alpha \gamma^r}{\alpha \gamma^r + \beta \mu^r}$. This definition is motivated by the infinite-dimensional generalizations of Ky Fan's inequality [14] on the log-concavity of the determinant of SPD matrices, as stated for \det_X in Theorem 1 in [1] and Theorem 5 in [2].

In the following, we show that the definition given in Eq. (52) is valid in the more general case $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_2(\mathcal{H})$. We first have the following results.

Proposition 2. Assume that $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_2(\mathcal{H})$. Let $\alpha > 0, \beta > 0$ be fixed. Let $p, q \in \mathbb{R}$ be such that $p\alpha(\gamma/\mu)^p = q\beta(\gamma/\mu)^{-q}$. Then for $\Lambda + \frac{\gamma}{\mu} I = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}$,

$$\frac{\alpha(\Lambda + \frac{\gamma}{\mu} I)^p + \beta(\Lambda + \frac{\gamma}{\mu} I)^{-q}}{\alpha + \beta} \in \mathcal{PC}_1(\mathcal{H}). \quad (53)$$

Proposition 3. Assume that $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_2(\mathcal{H})$. Let $\Lambda + \frac{\gamma}{\mu} I = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}$. Let $\alpha > 0, \beta > 0$ be fixed. Let $p, q \in \mathbb{R}$ be such that $p\alpha(\gamma/\mu)^p = q\beta(\gamma/\mu)^{-q}$. Then

$$\frac{\alpha[(A + \gamma I)(B + \mu I)^{-1}]^p + \beta[(A + \gamma I)(B + \mu I)^{-1}]^{-q}}{\alpha + \beta} \in \text{Tr}_X(\mathcal{H}). \quad (54)$$

Furthermore,

$$\begin{aligned} & \det_X \left[\frac{\alpha[(A + \gamma I)(B + \mu I)^{-1}]^p + \beta[(A + \gamma I)(B + \mu I)^{-1}]^{-q}}{\alpha + \beta} \right] \\ &= \det_X \left[\frac{\alpha(\Lambda + \frac{\gamma}{\mu} I)^p + \beta(\Lambda + \frac{\gamma}{\mu} I)^{-q}}{\alpha + \beta} \right]. \end{aligned} \quad (55)$$

Motivated by Eq. (52) and Propositions 2 and 3, the following is our definition of the Alpha-Beta Log-Determinant divergences on $\mathcal{PC}_2(\mathcal{H})$.

Definition 2 (Alpha-Beta Log-Determinant divergences between positive definite Hilbert-Schmidt operators). Assume that $\dim(\mathcal{H}) = \infty$. Let $\alpha > 0, \beta > 0$ be fixed. Let $r \in \mathbb{R}, r \neq 0$ be fixed. For $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_2(\mathcal{H})$, the (α, β) -Log-Det divergence $D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)]$ is defined to be

$$\begin{aligned} & D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] \\ &= \frac{1}{\alpha\beta} \log \left[\left(\frac{\gamma}{\mu} \right)^{r(\delta - \frac{\alpha}{\alpha + \beta})} \det_X \left(\frac{\alpha(\Lambda + \frac{\gamma}{\mu} I)^{r(1-\delta)} + \beta(\Lambda + \frac{\gamma}{\mu} I)^{-r\delta}}{\alpha + \beta} \right) \right], \end{aligned} \quad (56)$$

where $\Lambda + \frac{\gamma}{\mu} I = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}$, $\delta = \frac{\alpha\gamma^r}{\alpha\gamma^r + \beta\mu^r}$. Equivalently,

$$\begin{aligned} & D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] \\ &= \frac{1}{\alpha\beta} \log \left[\left(\frac{\gamma}{\mu} \right)^{r(\delta - \frac{\alpha}{\alpha + \beta})} \det_X \left(\frac{\alpha(Z + \frac{\gamma}{\mu} I)^{r(1-\delta)} + \beta(Z + \frac{\gamma}{\mu} I)^{-r\delta}}{\alpha + \beta} \right) \right], \end{aligned} \quad (57)$$

where $Z + \frac{\gamma}{\mu} I = (A + \gamma I)(B + \mu I)^{-1}$.

While Definition 2 is stated using the extended Fredholm determinant \det_X , the limiting cases $(\alpha > 0, \beta = 0)$ and $(\alpha = 0, \beta > 0)$ both require the concept of the extended Hilbert-Carleman determinant \det_{2X} .

Theorem 1 (Limiting case $\alpha > 0, \beta \rightarrow 0$). Let $\alpha > 0$ be fixed. Assume that $r = r(\beta)$ is smooth, with $r(0) = r(\beta = 0)$. Then

$$\begin{aligned} \lim_{\beta \rightarrow 0} D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] &= \frac{1}{\alpha^2} \left[\left(\frac{\mu}{\gamma} \right)^{r(0)} - 1 \right] \left(1 + r(0) \log \frac{\mu}{\gamma} \right) \\ &\quad - \frac{1}{\alpha^2} \left(\frac{\mu}{\gamma} \right)^{r(0)} \log \det_{2X}([(A + \gamma I)^{-1}(B + \mu I)]^{r(0)}). \end{aligned} \quad (58)$$

Theorem 2 (Limiting case $\alpha \rightarrow 0, \beta > 0$). *Let $\beta > 0$ be fixed. Assume that $r = r(\alpha)$ is smooth, with $r(0) = r(\alpha = 0)$. Then*

$$\begin{aligned} \lim_{\alpha \rightarrow 0} D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] &= \frac{1}{\beta^2} \left[\left(\frac{\gamma}{\mu} \right)^{r(0)} - 1 \right] \left(1 + r(0) \log \frac{\gamma}{\mu} \right) \\ &\quad - \frac{1}{\beta^2} \left(\frac{\gamma}{\mu} \right)^{r(0)} \log \det_{2X}([(B + \mu I)^{-1}(A + \gamma I)]^{r(0)}). \end{aligned} \quad (59)$$

Motivated by Theorems 1 and 2, the following is our definition of $D_r^{(\alpha, 0)}[(A + \gamma I), (B + \mu I)]$ and $D_r^{(0, \beta)}[(A + \gamma I), (B + \mu I)]$, $\alpha > 0, \beta > 0$.

Definition 3 (Limiting cases). *Let $\alpha, \beta > 0$ be fixed. Let $r \in \mathbb{R}, r \neq 0$ be fixed. For $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_2(\mathcal{H})$, the divergence $D_r^{(\alpha, 0)}[(A + \gamma I), (B + \mu I)]$ is defined to be*

$$\begin{aligned} D_r^{(\alpha, 0)}[(A + \gamma I), (B + \mu I)] &= \frac{1}{\alpha^2} \left[\left(\frac{\mu}{\gamma} \right)^r - 1 \right] \left(1 + r \log \frac{\mu}{\gamma} \right) \\ &\quad - \frac{1}{\alpha^2} \left(\frac{\mu}{\gamma} \right)^r \log \det_{2X}([(A + \gamma I)^{-1}(B + \mu I)]^r). \end{aligned} \quad (60)$$

Similarly, the divergence $D_r^{(0, \beta)}[(A + \gamma I), (B + \mu I)]$ is defined to be

$$\begin{aligned} D_r^{(0, \beta)}[(A + \gamma I), (B + \mu I)] &= \frac{1}{\beta^2} \left[\left(\frac{\gamma}{\mu} \right)^r - 1 \right] \left(1 + r \log \frac{\gamma}{\mu} \right) \\ &\quad - \frac{1}{\beta^2} \left(\frac{\gamma}{\mu} \right)^r \log \det_{2X}([(B + \mu I)^{-1}(A + \gamma I)]^r). \end{aligned} \quad (61)$$

For the case $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_1(\mathcal{H})$, from Definition 3, we recover the formulation stated in Definition 2 in [2], as follows.

Corollary 2. *Let $\alpha, \beta > 0$ be fixed. Let $r \in \mathbb{R}, r \neq 0$ be fixed. Assume that $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_1(\mathcal{H})$. Then in Definition 3,*

$$\begin{aligned} D_r^{(\alpha, 0)}[(A + \gamma I), (B + \mu I)] &= \frac{r}{\alpha^2} \left[\left(\frac{\mu}{\gamma} \right)^r - 1 \right] \log \frac{\mu}{\gamma} \\ &\quad - \frac{1}{\alpha^2} \left(\frac{\mu}{\gamma} \right)^r \log \det_X[(A + \gamma I)^{-1}(B + \mu I)]^r \\ &\quad + \frac{1}{\alpha^2} \text{tr}_X([(A + \gamma I)^{-1}(B + \mu I)]^r - I). \end{aligned} \quad (62)$$

$$\begin{aligned}
D_r^{(0,\beta)}[(A + \gamma I), (B + \mu I)] &= \frac{r}{\beta^2} \left[\left(\frac{\gamma}{\mu} \right)^r - 1 \right] \log \frac{\gamma}{\mu} \\
&\quad - \frac{1}{\beta^2} \left(\frac{\gamma}{\mu} \right)^r \log \det_X[(B + \mu I)^{-1}(A + \gamma I)]^r \\
&\quad + \frac{1}{\beta^2} \text{tr}_X[(B + \mu I)^{-1}(A + \gamma I)]^r - I. \tag{63}
\end{aligned}$$

The following is the generalization of Theorem 9 in [2] to positive definite Hilbert-Schmidt operators.

Theorem 3 (Limiting case $(0, 0)$). *Assume that $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_2(\mathcal{H})$. Assume that $r = r(\alpha)$ is smooth, with $r(0) = 0$, $r'(0) \neq 0$, and $r(\alpha) \neq 0$ for $\alpha \neq 0$. Then*

$$\lim_{\alpha \rightarrow 0} D_r^{(\alpha, \alpha)}[(A + \gamma I), (B + \mu I)] = \frac{[r'(0)]^2}{8} d_{\text{aiHS}}^2[(A + \gamma I), (B + \mu I)]. \tag{64}$$

In particular, for $r = 2\alpha$,

$$\lim_{\alpha \rightarrow 0} D_{2\alpha}^{(\alpha, \alpha)}[(A + \gamma I), (B + \mu I)] = \frac{1}{2} d_{\text{aiHS}}^2[(A + \gamma I), (B + \mu I)]. \tag{65}$$

The following is the generalization of Theorem 3 in [2] to positive definite Hilbert-Schmidt operators.

Theorem 4 (Symmetric divergences). *The parametrized family $D_{2\alpha}^{(\alpha, \alpha)}[(A + \gamma I), (B + \mu I)]$, $\alpha \geq 0$, is a family of symmetric divergences on $\mathcal{PC}_2(\mathcal{H})$, with $\alpha = 0$ corresponding to the infinite-dimensional affine-invariant Riemannian distance above and $\alpha = 1/2$ corresponding to the infinite-dimensional symmetric Stein divergence, which is given by $\frac{1}{4} d_{\log \det}^0[(A + \gamma I), (B + \mu I)]$.*

6. Properties of the Log-Determinant divergences

The following results establish several important properties of $D_r^{(\alpha, \beta)}$ as defined above, which generalize those from both the finite-dimensional setting [9, 3] and the infinite-dimensional Alpha Log-Det divergences [1] and Alpha-Beta Log-Det divergences [2] for positive definite trace class operators.

In the following theorems, $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_2(\mathcal{H})$.

Theorem 5 (Positivity).

$$D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] \geq 0, \quad (66)$$

$$D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] = 0 \iff A = B, \gamma = \mu. \quad (67)$$

Theorem 6 (Dual symmetry).

$$D_r^{(\beta, \alpha)}[(B + \mu I), (A + \gamma I)] = D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)]. \quad (68)$$

In particular, for $\beta = \alpha$, we have

$$D_r^{(\alpha, \alpha)}[(B + \mu I), (A + \gamma I)] = D_r^{(\alpha, \alpha)}[(A + \gamma I), (B + \mu I)]. \quad (69)$$

Theorem 7 (Dual invariance under inversion).

$$D_r^{(\alpha, \beta)}[(A + \gamma I)^{-1}, (B + \mu I)^{-1}] = D_{-r}^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] \quad (70)$$

Theorem 8 (Affine invariance). For any $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_2(\mathcal{H})$ and any invertible $(C + \nu I) \in \text{HS}_X(\mathcal{H})$, $\nu \neq 0$,

$$\begin{aligned} & D_r^{(\alpha, \beta)}[(C + \nu I)(A + \gamma I)(C + \nu I)^*, (C + \nu I)(B + \mu I)(C + \nu I)^*] \\ &= D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)]. \end{aligned} \quad (71)$$

Theorem 9 (Invariance under unitary transformations). For any $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_2(\mathcal{H})$ and any $C \in \mathcal{L}(\mathcal{H})$, with $CC^* = C^*C = I$,

$$D_r^{(\alpha, \beta)}[C(A + \gamma I)C^*, C(B + \mu I)C^*] = D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)]. \quad (72)$$

7. Proofs of main results

7.1. Proofs of the properties of the extended Hilbert-Carleman determinant

Proof of Lemma 1. By definition of the Hilbert-Carleman determinant and the assumption that $I + A > 0$, we have

$$\det_2(I + A) = \det[(I + A) \exp(-A)] = \prod_{k=1}^{\infty} [(1 + \lambda_k) \exp(-\lambda_k)] > 0.$$

Thus $\log \det_2(I + A)$ is well-defined and finite, and is given by the series

$$\log \det_2(I + A) = \sum_{k=1}^{\infty} [\log(1 + \lambda_k) - \lambda_k],$$

which necessarily has a finite value.

For the second statement, consider the function $f(x) = \log(1 + x) - x$ for $x > -1$. We have $f'(x) = -\frac{x}{1+x}$, with $f'(x) > 0$ for $-1 < x < 0$ and $f'(x) < 0$ for $x > 0$. Thus f has a unique global maximum $f_{\max} = f(0) = 0$. Thus $\forall k \in \mathbb{N}$,

$$\log(1 + \lambda_k) - \lambda_k \leq 0, \quad \text{with equality if and only } \lambda_k = 0.$$

It then follows that $\log(I + A) \leq 0$, with equality if and only $\lambda_k = 0 \forall k \in \mathbb{N}$, that is if and only if $A = 0$. \square

Proof of Lemma 2. We make use of the result that $R_2(A) = (I + A) \exp(-A) - I \in \text{Tr}(\mathcal{H})$ for $A \in \text{HS}(\mathcal{H})$. Thus

$$R_{2,\gamma}(A) = (A + \gamma I) \exp(-A/\gamma) - \gamma I = \gamma \left[\left(\frac{A}{\gamma} + I \right) \exp(-A/\gamma) - I \right] \in \text{Tr}(\mathcal{H}),$$

and hence $R_{2,\gamma}(A) + \gamma I = (A + \gamma I) \exp(-A/\gamma) \in \text{Tr}_X(\mathcal{H})$. Since $R_{2,\gamma}(A) \in \text{Tr}(\mathcal{H})$, the infinite product

$$\prod_{k=1}^{\infty} [(\lambda_k + \gamma) \exp(-\lambda_k/\gamma) - \gamma + 1] = \det[R_{2,\gamma}(A) + I]$$

converges to a finite value. \square

Proof of Lemma 3 (Factorization Rule). We have $\left(\frac{A}{\gamma} + I \right) \exp(-A/\gamma) - I \in \text{Tr}(\mathcal{H})$ and thus for the operator

$$(A + \gamma I) \exp(-A/\gamma) = \gamma \left[\left(\frac{A}{\gamma} + I \right) \exp(-A/\gamma) - I \right] + \gamma I \in \text{Tr}_X(\mathcal{H}),$$

its extended Fredholm determinant is given by

$$\det_X[(A + \gamma I) \exp(-A/\gamma)] = \gamma \det \left[\left(\frac{A}{\gamma} + I \right) \exp(-A/\gamma) \right] = \gamma \det_2 \left(\frac{A}{\gamma} + I \right).$$

This completes the proof. \square

Proof of Lemma 4 (Similarity Invariant). Since $\text{HS}(\mathcal{H})$ is a two-sided ideal in $\mathcal{L}(\mathcal{H})$, we have $CAC^{-1} \in \text{HS}(\mathcal{H})$. Thus

$$C(A + \gamma I)C^{-1} = CAC^{-1} + \gamma I \in \text{HS}_X(\mathcal{H}).$$

By definition of the extended Hilbert-Carleman determinant, we have

$$\begin{aligned} \det_{2X}[C(A + \gamma I)C^{-1}] &= \det_X[C(A + \gamma I)C^{-1} \exp(-CAC^{-1}/\gamma)] \\ &= \det_X[C(A + \gamma I)C^{-1}(C \exp(-A/\gamma)C^{-1})] = \det_X[C(A + \gamma I) \exp(-A/\gamma)C^{-1}] \\ &= \det_X[(A + \gamma I) \exp(-A/\gamma)] \quad \text{by Eq. (34)} \\ &= \det_{2X}(A + \gamma I). \end{aligned}$$

This completes the proof. \square

Proof of Lemma 5 (Commutativity). Consider the first assumption, that is $(A + \gamma I), (B + \mu I) \in \mathcal{P}\mathcal{C}_2(\mathcal{H})$. We write $(A + \gamma I)(B + \mu I)$ and $(B + \mu I)(A + \gamma I)$ as

$$\begin{aligned} (A + \gamma I)(B + \mu I) &= (A + \gamma I)^{1/2}[(A + \gamma I)^{1/2}(B + \mu I)(A + \gamma I)^{1/2}](A + \gamma I)^{-1/2}, \\ (B + \mu I)(A + \gamma I) &= (A + \gamma I)^{-1/2}[(A + \gamma I)^{1/2}(B + \mu I)(A + \gamma I)^{1/2}](A + \gamma I)^{1/2}. \end{aligned}$$

By Lemma 4, we then have

$$\begin{aligned} \det_{2X}[(A + \gamma I)(B + \mu I)] &= \det_{2X}[(A + \gamma I)^{1/2}(B + \mu I)(A + \gamma I)^{1/2}] \\ &= \det_{2X}[(B + \mu I)(A + \gamma I)]. \end{aligned}$$

The third statement is proved similarly.

Under the second assumption, that is $(A + \gamma I), (B + \mu I) \in \text{Tr}_X(\mathcal{H})$, $\gamma \neq 0, \mu \neq 0$, we have by definition

$$\begin{aligned} \det_{2X}[(A + \gamma I)(B + \mu I)] &= \det_X[(A + \gamma I)(B + \mu I)] \exp\left(-\frac{\text{tr}[\mu A + \gamma B + AB]}{\gamma\mu}\right) \\ &= \det_X[(B + \mu I)(A + \gamma I)] \exp\left(-\frac{\text{tr}[\mu A + \gamma B + BA]}{\gamma\mu}\right) \\ &= \det_{2X}[(B + \mu I)(A + \gamma I)]. \end{aligned}$$

Here we have made use of the properties $\det_X[(A + \gamma I)(B + \mu I)] = \det_X(A + \gamma I)\det_X(B + \mu I) = \det_X[(B + \mu I)(A + \gamma I)]$ and the commutativity of the trace, namely $\text{tr}(AB) = \text{tr}(BA)$. This completes the proof. \square

Proof of Lemma 6. We rewrite $(A + \gamma I)(B + \mu I)^{-1}$ as

$$\begin{aligned} & (A + \gamma I)(B + \mu I)^{-1} \\ &= (B + \mu I)^{1/2}[(B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}](B + \mu I)^{-1/2} \\ &= (B + \mu I)^{1/2}(\Lambda + \nu I)(B + \mu I)^{-1/2}. \end{aligned}$$

Similarly,

$$(B + \mu I)^{-1}(A + \gamma I) = (B + \mu I)^{-1/2}(\Lambda + \nu I)(B + \mu I)^{1/2}.$$

By definition of the power function, we then have for any $\alpha \in \mathbb{R}$

$$\begin{aligned} [(A + \gamma I)(B + \mu I)^{-1}]^\alpha &= (B + \mu I)^{1/2}(\Lambda + \nu I)^\alpha(B + \mu I)^{-1/2}, \\ [(B + \mu I)^{-1}(A + \gamma I)]^\alpha &= (B + \mu I)^{-1/2}(\Lambda + \nu I)^\alpha(B + \mu I)^{1/2}. \end{aligned}$$

Thus by Lemma 4, we obtain

$$\begin{aligned} \det_{2X}([(A + \gamma I)(B + \mu I)^{-1}]^\alpha) &= \det_{2X}[(\Lambda + \nu I)^\alpha] \\ &= \det_{2X}[(B + \mu I)^{-1}(A + \gamma I)^\alpha]. \end{aligned}$$

This completes the proof. \square

Lemma 9. Let $r \neq 0$ be fixed. The function $f(x) = x^r - 1 - r \log(x)$ for $x > 0$ has a unique global minimum $f_{\min} = f(1) = 0$. In other words, $f(x) \geq 0 \forall x > 0$, with equality if and only if $x = 1$.

Proof of Lemma 9. We have $f'(x) = \frac{r(x^r - 1)}{x}$. When $r > 0$, we have $x^r < 1$ for $0 < x < 1$ and $x^r > 1$ for $x > 1$. When $r < 0$, we have $x^r > 1$ for $0 < x < 1$ and $x^r < 1$ for $x > 1$. Thus, for all $r \neq 0$, we have $f'(x) < 0$ when $0 < x < 1$ and $f'(x) > 0$ when $x > 1$. Hence f has a unique global minimum $f_{\min} = f(1) = 0$. \square

Proof of Lemma 7. By Proposition 2 in [15], we have $\log(I + A) \in \text{HS}(\mathcal{H})$ for $(I + A) \in \mathcal{PC}_2(\mathcal{H})$. By definition of the power function, we have

$$(I + A)^\alpha = \exp[\alpha \log(I + A)] = I + \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} [\log(I + A)]^j.$$

Since $\text{HS}(\mathcal{H})$ is a Banach algebra under the Hilbert-Schmidt norm, we then have

$$\begin{aligned} \|(I + A)^\alpha - I\|_{\text{HS}} &= \left\| \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} [\log(I + A)]^j \right\|_{\text{HS}} \leq \sum_{j=1}^{\infty} \frac{|\alpha|^j}{j!} \|\log(I + A)\|_{\text{HS}}^j \\ &= \exp(|\alpha| \|\log(I + A)\|_{\text{HS}}) - 1 < \infty. \end{aligned}$$

Thus $(I + A)^\alpha - I \in \text{HS}(\mathcal{H})$. By definition of the Hilbert-Carleman determinant, we then have

$$\begin{aligned} \det_2[(I + A)^\alpha] &= \det[(I + A)^\alpha \exp(-[(I + A)^\alpha - I])] \\ &= \prod_{k=1}^{\infty} (1 + \lambda_k)^\alpha \exp[1 - (1 + \lambda_k)^\alpha] < \infty. \end{aligned}$$

Thus the following quantity is well-defined and finite

$$\log \det_2[(I + A)^\alpha] = \sum_{k=1}^{\infty} [\alpha \log(1 + \lambda_k) + 1 - (1 + \lambda_k)^\alpha].$$

The statements for the case $I + A \in \mathcal{P}\mathcal{C}_1(\mathcal{H})$ are then obvious from the above series expansions.

By Lemma 9, we have $\forall k \in \mathbb{N}$,

$$\alpha \log(1 + \lambda_k) + 1 - (1 + \lambda_k)^\alpha \leq 0,$$

with equality if and only if $\lambda_k = 0$. Thus it follows that

$$\log \det_2[(I + A)^\alpha] \leq 0,$$

with equality if and only if $\lambda_k = 0 \ \forall k \in \mathbb{N}$, that is if and only if $A = 0$ (by the assumption that $I + A > 0$). \square

Proof of Lemma 8. By definition of the power function, we have

$$\begin{aligned} (A + \gamma I)^\alpha &= \exp[\alpha \log(A + \gamma I)] = \exp \left[(\alpha \log \gamma) I + \alpha \log \left(\frac{A}{\gamma} + I \right) \right] \\ &= \gamma^\alpha \left(\frac{A}{\gamma} + I \right)^\alpha = \gamma^\alpha \left[\left(\frac{A}{\gamma} + I \right)^\alpha - I \right] + \gamma^\alpha I, \end{aligned}$$

where $\left[\left(\frac{A}{\gamma} + I \right)^\alpha - I \right] \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ by Lemma 7. Thus it follows that $(A + \gamma I)^\alpha - \gamma^\alpha I \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$. Therefore, the extended Hilbert-Carleman

determinant of $(A + \gamma I)^\alpha$ is well-defined and finite. By the Factorization Rule (Lemma 3) and Lemma 7, we have

$$\begin{aligned}\det_{2X}[(A + \gamma I)^\alpha] &= \gamma^\alpha \det_2 \left[\left(\frac{A}{\gamma} + I \right)^\alpha \right] \\ &= \gamma^\alpha \prod_{k=1}^{\infty} \left(1 + \frac{\lambda_k}{\gamma} \right)^\alpha \exp \left[1 - \left(1 + \frac{\lambda_k}{\gamma} \right)^\alpha \right] < \infty.\end{aligned}$$

Consequently, the following quantity is also finite

$$\log \det_{2X}[(A + \gamma I)^\alpha] = \alpha \log \gamma + \sum_{k=1}^{\infty} \left[\alpha \log \left(1 + \frac{\lambda_k}{\gamma} \right) + 1 - \left(1 + \frac{\lambda_k}{\gamma} \right)^\alpha \right].$$

If $A + \gamma I \in \mathcal{P}\mathcal{C}_1(\mathcal{H})$, then $(\frac{A}{\gamma} + I)^\alpha - I \in \text{Tr}(\mathcal{H})$ (see Lemma 6 in [2], or by using a similar argument as in Lemma 7). Thus the following infinite product and series

$$\begin{aligned}\det \left(\frac{A}{\gamma} + I \right)^\alpha &= \prod_{k=1}^{\infty} \left(\frac{\lambda_k}{\gamma} + 1 \right)^\alpha, \\ \alpha \log \det \left(\frac{A}{\gamma} + I \right) &= \alpha \sum_{k=1}^{\infty} \log \left(\frac{\lambda_k}{\gamma} + 1 \right), \\ \text{tr} \left[\left(\frac{A}{\gamma} + I \right)^\alpha - I \right] &= \sum_{k=1}^{\infty} \left[\left(1 + \frac{\lambda_k}{\gamma} \right)^\alpha - 1 \right]\end{aligned}$$

converge to finite values. These give the last statements of the lemma. \square

7.2. Proofs for the definition of the Log-Determinant divergences

Lemma 10. *Let $\alpha > 0, \beta > 0, \gamma > 0$ be fixed. Let $p, q \in \mathbb{R}$ be such that $p\alpha\gamma^p = q\beta\gamma^{-q}$. Then*

$$\lim_{x \rightarrow 0} \frac{1 - \frac{\alpha\gamma^p(1+x)^p + \beta\gamma^{-q}(1+x)^{-q}}{\alpha\gamma^p + \beta\gamma^{-q}}}{x^2} = -\frac{p(p-1)\alpha\gamma^p + q(q+1)\beta\gamma^{-q}}{2(\alpha\gamma^p + \beta\gamma^{-q})}. \quad (73)$$

Proof of Lemma 10. Since the limit has the form $\frac{0}{0}$, by L'Hopital's rule, we have

$$\begin{aligned}&\lim_{x \rightarrow 0} \frac{1 - \frac{\alpha\gamma^p(1+x)^p + \beta\gamma^{-q}(1+x)^{-q}}{\alpha\gamma^p + \beta\gamma^{-q}}}{x^2} \\ &= -\frac{1}{2(\alpha\gamma^p + \beta\gamma^{-q})} \lim_{x \rightarrow 0} \frac{p\alpha\gamma^p(1+x)^{p-1} - q\beta\gamma^{-q}(1+x)^{-q-1}}{x}.\end{aligned}$$

By assumption, we have $p\alpha\gamma^p = q\beta\gamma^{-q}$, so that the previous limit also has the form $\frac{0}{0}$. Applying L'Hopital's rule one more time, we obtain

$$\begin{aligned} & -\frac{1}{2(\alpha\gamma^p + \beta\gamma^{-q})} \lim_{x \rightarrow 0} [p(p-1)\alpha\gamma^p(1+x)^{p-2} + q(q+1)\beta\gamma^{-q}(1+x)^{-q-2}] \\ & = -\frac{p(p-1)\alpha\gamma^p + q(q+1)\beta\gamma^{-q}}{2(\alpha\gamma^p + \beta\gamma^{-q})}. \end{aligned}$$

This completes the proof. \square

Corollary 3. *Let $A \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ be such that $I + A > 0$. Let $\alpha > 0, \beta > 0, \gamma > 0$ be fixed. Let $p, q \in \mathbb{R}$ be such that $p\alpha\gamma^p = q\beta\gamma^{-q}$. Then*

$$I - \frac{\alpha\gamma^p(I + A)^p + \beta\gamma^{-q}(I + A)^{-q}}{\alpha\gamma^p + \beta\gamma^{-q}} \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}). \quad (74)$$

Proof of Corollary 3. Let $\{\lambda_k\}_{k=1}^\infty$ denote the eigenvalues of A then $\lim_{k \rightarrow \infty} \lambda_k = 0$. By Lemma 10, we have

$$\lim_{k \rightarrow \infty} \frac{1 - \frac{\alpha\gamma^p(1+\lambda_k)^p + \beta\gamma^{-q}(1+\lambda_k)^{-q}}{\alpha\gamma^p + \beta\gamma^{-q}}}{\lambda_k^2} = -\frac{p(p-1)\alpha\gamma^p + q(q+1)\beta\gamma^{-q}}{2(\alpha\gamma^p + \beta\gamma^{-q})}.$$

This implies that there exists a constant $C > 0$, independent of k , and a number $N = N(C) \in \mathbb{N}$, such that

$$\left| 1 - \frac{\alpha\gamma^p(1 + \lambda_k)^p + \beta\gamma^{-q}(1 + \lambda_k)^{-q}}{\alpha\gamma^p + \beta\gamma^{-q}} \right| \leq C\lambda_k^2 \quad \forall k \geq N.$$

Since $\sum_{k=1}^\infty \lambda_k^2 < \infty$ by assumption, it then follows that

$$\sum_{k=1}^\infty \left| 1 - \frac{\alpha\gamma^p(1 + \lambda_k)^p + \beta\gamma^{-q}(1 + \lambda_k)^{-q}}{\alpha\gamma^p + \beta\gamma^{-q}} \right| < \infty,$$

which gives us the desired result. \square

Proof of Proposition 2. Since $(A + \gamma I), (B + \mu I) \in \mathcal{P}\mathcal{C}_2(\mathcal{H})$, we have $\Lambda + \frac{\gamma}{\mu}I = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2} \in \mathcal{P}\mathcal{C}_2(\mathcal{H})$, with $\Lambda \in \text{HS}(\mathcal{H})$. Thus it is obvious that $\frac{\alpha(\Lambda + \frac{\gamma}{\mu}I)^p + \beta(\Lambda + \frac{\gamma}{\mu}I)^{-q}}{\alpha + \beta}$ is also positive definite. Let us show that it is an

extended trace class operator. Consider the expansion

$$\begin{aligned}
& \frac{\alpha(\Lambda + \frac{\gamma}{\mu}I)^p + \beta(\Lambda + \frac{\gamma}{\mu}I)^{-q}}{\alpha + \beta} \\
&= \frac{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}}{\alpha + \beta} \frac{\alpha(\frac{\gamma}{\mu})^p(\frac{\mu}{\gamma}\Lambda + I)^p + \beta(\frac{\gamma}{\mu})^{-q}(\frac{\mu}{\gamma}\Lambda + I)^{-q}}{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}} \\
&= \frac{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}}{\alpha + \beta} \left[I - \left(I - \frac{\alpha(\frac{\gamma}{\mu})^p(\frac{\mu}{\gamma}\Lambda + I)^p + \beta(\frac{\gamma}{\mu})^{-q}(\frac{\mu}{\gamma}\Lambda + I)^{-q}}{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}} \right) \right].
\end{aligned}$$

By Corollary 3, we have $\left(I - \frac{\alpha(\frac{\gamma}{\mu})^p(\frac{\mu}{\gamma}\Lambda + I)^p + \beta(\frac{\gamma}{\mu})^{-q}(\frac{\mu}{\gamma}\Lambda + I)^{-q}}{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}} \right) \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$.

Thus it follows that $\frac{\alpha(\Lambda + \frac{\gamma}{\mu}I)^p + \beta(\Lambda + \frac{\gamma}{\mu}I)^{-q}}{\alpha + \beta} \in \text{Tr}_X(\mathcal{H})$. This completes the proof. \square

Proof of Proposition 3. By Proposition 2, we have

$$\frac{\alpha(\Lambda + \frac{\gamma}{\mu}I)^p + \beta(\Lambda + \frac{\gamma}{\mu}I)^{-q}}{\alpha + \beta} \in \mathcal{PC}_1(\mathcal{H}).$$

Thus its extended Fredholm determinant \det_X is well-defined and finite.

By Proposition 1, we have for any $p \in \mathbb{R}$,

$$[(A + \gamma I)(B + \mu I)^{-1}]^p = (B + \mu I)^{1/2}(\Lambda + \frac{\gamma}{\mu}I)^p(B + \mu I)^{-1/2} \in \text{HS}_X(\mathcal{H}).$$

Thus it follows that

$$\begin{aligned}
& \frac{\alpha[(A + \gamma I)(B + \mu I)^{-1}]^p + \beta[(A + \gamma I)(B + \mu I)^{-1}]^{-q}}{\alpha + \beta} \\
&= (B + \mu I)^{1/2} \left[\frac{\alpha(\Lambda + \frac{\gamma}{\mu}I)^p + \beta(\Lambda + \frac{\gamma}{\mu}I)^{-q}}{\alpha + \beta} \right] (B + \mu I)^{-1/2} \in \text{Tr}_X(\mathcal{H}).
\end{aligned}$$

Thus by Eq. (34), we obtain

$$\begin{aligned}
& \det_X \left[\frac{\alpha[(A + \gamma I)(B + \mu I)^{-1}]^p + \beta[(A + \gamma I)(B + \mu I)^{-1}]^{-q}}{\alpha + \beta} \right] \\
&= \det_X \left[\frac{\alpha(\Lambda + \frac{\gamma}{\mu}I)^p + \beta(\Lambda + \frac{\gamma}{\mu}I)^{-q}}{\alpha + \beta} \right].
\end{aligned}$$

This completes the proof. \square

Proof of Theorems 1 and 2. Let $\{\lambda_j\}_{j=1}^\infty$ be the eigenvalues of Λ . By Theorem 8 in

[2], we have the following expansion

$$\begin{aligned}
D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] &= \frac{r(\delta - \frac{\alpha}{\alpha + \beta})}{\alpha\beta} \log\left(\frac{\gamma}{\mu}\right) + \frac{1}{\alpha\beta} \log\left(\frac{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}}{\alpha + \beta}\right) \\
&\quad + \frac{1}{\alpha\beta} \log \det\left(\frac{\alpha(\Lambda + \frac{\gamma}{\mu} I)^p + \beta(\Lambda + \frac{\gamma}{\mu} I)^{-q}}{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}}\right) \\
&= \frac{r(\delta - \frac{\alpha}{\alpha + \beta})}{\alpha\beta} \log\left(\frac{\gamma}{\mu}\right) + \frac{1}{\alpha\beta} \log\left(\frac{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}}{\alpha + \beta}\right) \\
&\quad + \frac{1}{\alpha\beta} \sum_{j=1}^{\infty} \log\left(\frac{\alpha(\lambda_j + \frac{\gamma}{\mu})^p + \beta(\lambda_j + \frac{\gamma}{\mu})^{-q}}{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}}\right),
\end{aligned}$$

where $p = p(\beta) = r(1 - \delta) = \frac{r\beta}{\alpha(\frac{\gamma}{\mu})^r + \beta}$, $q = q(\beta) = r\delta = \frac{r\alpha(\frac{\gamma}{\mu})^r}{\alpha(\frac{\gamma}{\mu})^r + \beta}$.

Let $\nu = \frac{\gamma}{\mu}$. By the same argument as in the proof of Theorem 11 in [2], we have

$$\begin{aligned}
\lim_{\beta \rightarrow 0} D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] &= \frac{1}{\alpha^2} [\nu^{-r(0)} + r(0) \log(\nu) - 1] \\
&\quad + \frac{1}{\alpha^2} \sum_{j=1}^{\infty} \left[\frac{r(0)}{\nu^{r(0)}} \log\left(\frac{\lambda_j}{\nu} + 1\right) + \frac{1}{(\lambda_j + \nu)^{r(0)}} - \frac{1}{\nu^{r(0)}} \right]. \tag{75}
\end{aligned}$$

By Lemma 8, we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} \left[\frac{r(0)}{\nu^{r(0)}} \log\left(\frac{\lambda_j}{\nu} + 1\right) + \frac{1}{(\lambda_j + \nu)^{r(0)}} - \frac{1}{\nu^{r(0)}} \right] \\
&= -\nu^{-r(0)} \sum_{j=1}^{\infty} \left[-r(0) \log\left(\frac{\lambda_j}{\nu} + 1\right) - \left(\frac{\lambda_j}{\nu} + 1\right)^{-r(0)} + 1 \right] \\
&= -\nu^{-r(0)} (\log \det_{2X}[(\Lambda + \nu I)^{-r(0)}] + r(0) \log \nu).
\end{aligned}$$

Combining this with Eq. (75), we obtain

$$\begin{aligned}
&\lim_{\beta \rightarrow 0} D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] \\
&= \frac{1}{\alpha^2} \left[(\nu^{-r(0)} - 1)(1 - r(0) \log \nu) - \nu^{-r(0)} \log \det_{2X}[(\Lambda + \nu I)^{-r(0)}] \right].
\end{aligned}$$

By Lemma 6, we have

$$\begin{aligned}
\det_{2X}[(\Lambda + \nu I)^{-r(0)}] &= \det_{2X}[(B + \mu I)^{-1}(A + \gamma I)]^{-r(0)} \\
&= \det_{2X}[(A + \gamma I)^{-1}(B + \mu I)]^{r(0)}.
\end{aligned}$$

Substituting this into the previous expression and $\nu = \frac{\gamma}{\mu}$, we obtain the final result.

By dual symmetry, we then obtain $\lim_{\alpha \rightarrow 0} D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)]$ via

$$\lim_{\alpha \rightarrow 0} D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] = \lim_{\alpha \rightarrow 0} D^{(\beta, \alpha)}[(B + \mu I), (A + \gamma I)].$$

This completes the proof. \square

Proof of Corollary 2. Let us prove the first statement, since the second one is entirely similar. It suffices to prove for $\alpha = 1$. For $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_1(\mathcal{H})$, we have $(\Lambda + \nu I) = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2} \in \mathcal{PC}_1(\mathcal{H})$. By Definition 3,

$$D_r^{(1,0)}[(A + \gamma I), (B + \mu I)] = (\nu^{-r} - 1)(1 - r \log \nu) - \nu^{-r} \log \det_{2X}[(\Lambda + \nu I)^{-r}].$$

By Lemma 8, we have

$$\begin{aligned} \nu^{-r} \log \det_{2X}[(\Lambda + \nu I)^{-r}] &= -r\nu^{-r} \log \det_X(\Lambda + \nu I) - \nu^{-r} \operatorname{tr} \left[\left(\frac{\Lambda}{\nu} + I \right)^{-r} - I \right] \\ &= -r\nu^{-r} \log \det_X(\Lambda + \nu I) - \operatorname{tr}[(\Lambda + \nu I)^{-r} - \nu^{-r} I]. \end{aligned}$$

It then follows that

$$\begin{aligned} &(\nu^{-r} - 1)(1 - r \log \nu) - \nu^{-r} \log \det_{2X}[(\Lambda + \nu I)^{-r}] \\ &= (\nu^{-r} - 1)(1 - r \log \nu) + r\nu^{-r} \log \det_X(\Lambda + \nu I) + \operatorname{tr}[(\Lambda + \nu I)^{-r} - \nu^{-r} I] \\ &= -r(\nu^{-r} - 1) \log \nu + \nu^{-r} \log \det_X(\Lambda + \nu I)^r + (\nu^{-r} - 1 + \operatorname{tr}[(\Lambda + \nu I)^{-r} - \nu^{-r} I]) \\ &= -r(\nu^{-r} - 1) \log \nu - \nu^{-r} \log \det_X(\Lambda + \nu I)^{-r} + \operatorname{tr}_X[(\Lambda + \nu I)^{-r} - I]. \end{aligned}$$

By Lemma 8 in [2], which states that for any $\alpha \in \mathbb{R}$,

$$\begin{aligned} \det_X[(A + \gamma I)(B + \mu I)^{-1}]^\alpha &= \det_X[(\Lambda + \nu I)^\alpha] = \det_X[(B + \mu I)^{-1}(A + \gamma I)]^\alpha, \\ \operatorname{tr}_X[(A + \gamma I)(B + \mu I)^{-1}]^\alpha &= \operatorname{tr}_X[(\Lambda + \nu I)^\alpha] = \operatorname{tr}_X[(B + \mu I)^{-1}(A + \gamma I)]^\alpha, \end{aligned}$$

we have

$$\begin{aligned} \det_X(\Lambda + \nu I)^{-r} &= \det_X[(B + \mu I)^{-1}(A + \gamma I)]^{-r} = \det_X[(A + \gamma I)^{-1}(B + \mu I)]^r, \\ \operatorname{tr}_X[(\Lambda + \nu I)^{-r}] &= \operatorname{tr}_X[(B + \mu I)^{-1}(A + \gamma I)]^{-r} = \operatorname{tr}_X[(A + \gamma I)^{-1}(B + \mu I)]^r. \end{aligned}$$

Combining these with the previous expression, replacing $\nu = \frac{\gamma}{\mu}$, we obtain

$$\begin{aligned} D_r^{(1,0)}[(A + \gamma I), (B + \mu I)] &= \left(\left(\frac{\mu}{\gamma} \right)^r - 1 \right) \log \frac{\mu}{\gamma} \\ &\quad - \left(\frac{\mu}{\gamma} \right)^r \log \det_X[(A + \gamma I)^{-1}(B + \mu I)]^r \\ &\quad + \operatorname{tr}_X([(A + \gamma I)^{-1}(B + \mu I)]^r - I). \end{aligned}$$

This completes the proof. \square

Proof of Theorem 3. The proof is identical to the proof in the setting $(A + \gamma I), (B + \mu I) \in \mathcal{PC}_1(\mathcal{H})$ (Theorem 9 in [2]). \square

Proof of Theorem 4. This follows from the dual symmetry in Theorem 6 and the limiting behavior in Theorem 3. \square

7.3. Proofs of the properties of the Log-Determinant divergences

For the proof on Theorem 5 on positivity, we first need the following technical results.

Lemma 11. Assume that $\gamma > 0, \alpha > 0, \beta > 0$ are fixed. Let $r \in \mathbb{R}, r \neq 0$ be fixed. Then for $\delta = \frac{\alpha\gamma^r}{\alpha\gamma^r + \beta}$, $p = r(1 - \delta)$, $q = r\delta$, we have

$$\frac{r(\delta - \frac{\alpha}{\alpha + \beta})}{\alpha\beta} \log \gamma + \frac{1}{\alpha\beta} \log \left(\frac{\alpha\gamma^p + \beta\gamma^{-q}}{\alpha + \beta} \right) \geq 0. \quad (76)$$

Equality happens if and only if $\gamma = 1$.

Proof of Lemma 11. By the strict concavity of the log function, we have

$$\log \left(\frac{\alpha\gamma^p + \beta\gamma^{-q}}{\alpha + \beta} \right) \geq \frac{(p\alpha - q\beta) \log \gamma}{\alpha + \beta},$$

with equality if and only if $\gamma^p = \gamma^{-q} \iff \gamma^{p+q} = \gamma^r = 1$. Since $\gamma > 0$ and $r \neq 0$, this happens if and only if $\gamma = 1$. Thus we have

$$\begin{aligned} &\frac{r(\delta - \frac{\alpha}{\alpha + \beta})}{\alpha\beta} \log \gamma + \frac{1}{\alpha\beta} \log \left(\frac{\alpha\gamma^p + \beta\gamma^{-q}}{\alpha + \beta} \right) \\ &\geq \frac{1}{\alpha\beta} \left[r\left(\delta - \frac{\alpha}{\alpha + \beta}\right) + \frac{p\alpha - q\beta}{\alpha + \beta} \right] \log \gamma \\ &= \frac{1}{\alpha\beta} \left[q - \frac{(p+q)\alpha}{\alpha + \beta} + \frac{p\alpha - q\beta}{\alpha + \beta} \right] \log \gamma = 0, \quad \text{since } r = p + q. \end{aligned}$$

Equality happens if and only if $\gamma = 1$. \square

Lemma 12. Assume that $\gamma > 0, \alpha > 0, \beta > 0$ are fixed. Let $r \in \mathbb{R}, r \neq 0$ be fixed. Assume that $\lambda \in \mathbb{R}$ is also fixed, such that $\lambda + \gamma > 0$. Then for $\delta = \frac{\alpha\gamma^r}{\alpha\gamma^r + \beta}$, $p = r(1 - \delta)$, $q = r\delta$,

$$\log \left(\frac{\alpha(\lambda + \gamma)^p + \beta(\lambda + \gamma)^{-q}}{\alpha\gamma^p + \beta\gamma^{-q}} \right) \geq 0. \quad (77)$$

Equality happens if and only if $\lambda = 0$.

Proof of Lemma 12. By the strict concavity of the log function, we have

$$\begin{aligned} & \log \left(\frac{\alpha(\lambda + \gamma)^p + \beta(\lambda + \gamma)^{-q}}{\alpha\gamma^p + \beta\gamma^{-q}} \right) \\ &= \log \left[\frac{\alpha\gamma^p}{\alpha\gamma^p + \beta\gamma^{-q}} \left(\frac{\lambda}{\gamma} + 1 \right)^p + \frac{\beta\gamma^{-q}}{\alpha\gamma^p + \beta\gamma^{-q}} \left(\frac{\lambda}{\gamma} + 1 \right)^{-q} \right] \\ &\geq \frac{p\alpha\gamma^p - q\beta\gamma^{-q}}{\alpha\gamma^p + \beta\gamma^{-q}} \log \left(\frac{\lambda}{\gamma} + 1 \right) = 0, \end{aligned}$$

since $p\alpha\gamma^p - q\beta\gamma^{-q} = 0$, as can be verified directly using the given hypothesis. Equality happens if and only if $(\frac{\lambda}{\gamma} + 1)^p = (\frac{\lambda}{\gamma} + 1)^{-q} \iff (\frac{\lambda}{\gamma} + 1)^{p+q} = (\frac{\lambda}{\gamma} + 1)^r = 1$. Since $\lambda + \gamma > 0, \gamma > 0$, and $r \neq 0$, this happens if and only if $\lambda = 0$. \square

Proof of Theorem 5 (Positivity). (a) The case $\alpha > 0, \beta > 0$.

Let $\Lambda + \frac{\gamma}{\mu}I = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}$. Let $\{\lambda_j\}_{j=1}^\infty$ be the eigenvalues of Λ . By Theorem 8 in [2], we have the expansion

$$\begin{aligned} D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] &= \frac{r(\delta - \frac{\alpha}{\alpha + \beta})}{\alpha\beta} \log \left(\frac{\gamma}{\mu} \right) + \frac{1}{\alpha\beta} \log \left(\frac{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}}{\alpha + \beta} \right) \\ &\quad + \frac{1}{\alpha\beta} \log \det \left(\frac{\alpha(\Lambda + \frac{\gamma}{\mu}I)^p + \beta(\Lambda + \frac{\gamma}{\mu}I)^{-q}}{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}} \right) \\ &= \frac{r(\delta - \frac{\alpha}{\alpha + \beta})}{\alpha\beta} \log \left(\frac{\gamma}{\mu} \right) + \frac{1}{\alpha\beta} \log \left(\frac{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}}{\alpha + \beta} \right) \\ &\quad + \frac{1}{\alpha\beta} \sum_{j=1}^\infty \log \left(\frac{\alpha(\lambda_j + \frac{\gamma}{\mu})^p + \beta(\lambda_j + \frac{\gamma}{\mu})^{-q}}{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}} \right), \end{aligned}$$

where $p = p(\beta) = r(1 - \delta) = \frac{r\beta}{\alpha(\frac{\gamma}{\mu})^r + \beta}$, $q = q(\beta) = r\delta = \frac{r\alpha(\frac{\gamma}{\mu})^r}{\alpha(\frac{\gamma}{\mu})^r + \beta}$.

By Lemma 11, we have

$$\frac{r(\delta - \frac{\alpha}{\alpha + \beta})}{\alpha\beta} \log \left(\frac{\gamma}{\mu} \right) + \frac{1}{\alpha\beta} \log \left(\frac{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}}{\alpha + \beta} \right) \geq 0,$$

with equality if and only if $\frac{\gamma}{\mu} = 1 \iff \gamma = \mu$.

By Lemma 12, we have $\forall j \in \mathbb{N}$,

$$\log \left(\frac{\alpha(\lambda_j + \frac{\gamma}{\mu})^p + \beta(\lambda_j + \frac{\gamma}{\mu})^{-q}}{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}} \right) \geq 0,$$

with equality if and only $\lambda_j = 0$.

Combining these two results with the previous expression for $D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)]$, we obtain

$$D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] \geq 0,$$

with equality if and only if $\gamma = \mu$ and $\lambda_j = 0 \forall j \in \mathbb{N}$, that is $\Lambda = 0$. This is equivalent to $(B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2} = I \iff (A + \gamma I) = (B + \mu I) \iff A = B, \gamma = \mu$, since $A, B \in \text{HS}(\mathcal{H})$ by assumption.

(b) The case $\alpha = 0, \beta > 0$.

Since the factor β^2 can be ignored, it suffices to consider the case $\beta = 1$. We have

$$\begin{aligned} D_r^{(0,1)}[(A + \gamma I), (B + \mu I)] &= \left[\left(\frac{\gamma}{\mu} \right)^r - 1 \right] \left(1 + r \log \frac{\gamma}{\mu} \right) \\ &\quad - \left(\frac{\gamma}{\mu} \right)^r \log \det_{2X}([(B + \mu I)^{-1}(A + \gamma I)]^r). \end{aligned}$$

By Lemma 6, we have for any $r \in \mathbb{R}$,

$$\det_{2X}([(B + \mu I)^{-1}(A + \gamma I)]^r) = \det_{2X} \left[\left(\Lambda + \frac{\gamma}{\mu} I \right)^r \right] = \det_{2X} \left[\left(\frac{\gamma}{\mu} \right)^r \left(\frac{\mu}{\gamma} \Lambda + I \right)^r \right].$$

By the Factorization Rule in Lemma 3, we then have

$$\det_{2X}([(B + \mu I)^{-1}(A + \gamma I)]^r) = \left(\frac{\gamma}{\mu} \right)^r \det_2 \left[\left(\frac{\mu}{\gamma} \Lambda + I \right)^r \right].$$

Combining this with the first expression for $D_r^{(0,1)}[(A + \gamma I), (B + \mu I)]$, we obtain

$$\begin{aligned} D_r^{(0,1)}[(A + \gamma I), (B + \mu I)] &= \left(\frac{\gamma}{\mu} \right)^r - 1 - r \log \frac{\gamma}{\mu} \\ &\quad - \left(\frac{\gamma}{\mu} \right)^r \log \det_2 \left[\left(\frac{\mu}{\gamma} \Lambda + I \right)^r \right]. \end{aligned}$$

By Lemma 7, we have

$$\log \det_2 \left[\left(\frac{\mu}{\gamma} \Lambda + I \right)^r \right] \leq 0, \text{ with equality if and only if } \Lambda = 0.$$

By Lemma 9, we have

$$\left(\frac{\gamma}{\mu}\right)^r - 1 - r \log \frac{\gamma}{\mu} \geq 0, \text{ with equality if and only if } \frac{\gamma}{\mu} = 1.$$

Together with the previous expression for $D_r^{(0,1)}[(A + \gamma I), (B + \mu I)]$, these imply

$$\det_{2X}([(B + \mu I)^{-1}(A + \gamma I)]^r) \geq 0,$$

with equality if and only if $\Lambda + \frac{\gamma}{\mu}I = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2} = I \iff A + \gamma I = B + \mu I \iff A = B, \gamma = \mu$.

(c) The case $\alpha > 0, \beta = 0$ follows from the previous case by dual symmetry. This completes the proof. \square

Proof of Theorem 6 (Dual symmetry). For the case $\alpha > 0, \beta > 0$, the proof is identical to that for the setting $(A + \gamma I), (B + \mu I) \in \mathcal{P}\mathcal{C}_1(\mathcal{H})$ (Theorem 13 in [2]). The cases $\alpha = 0, \beta > 0$ and $\alpha > 0, \beta = 0$ are obvious from Eqs. (60) and (61). \square

Proof of Theorem 7 (Dual invariance under inversion). For the case $\alpha > 0, \beta > 0$, the proof is identical to that for the setting $(A + \gamma I), (B + \mu I) \in \mathcal{P}\mathcal{C}_1(\mathcal{H})$ (Theorem 14 in [2]).

Consider the case $\alpha = 0, \beta > 0$ (the case $\alpha > 0, \beta = 0$ follows from dual symmetry). It suffices to consider $\beta = 1$. We have

$$(A + \gamma I)^{-1} = \frac{1}{\gamma}I - \frac{A}{\gamma}(A + \gamma I)^{-1}, \quad (B + \mu I)^{-1} = \frac{1}{\mu}I - \frac{B}{\mu}(B + \mu I)^{-1}.$$

By Eq. (61), we have

$$\begin{aligned} D_r^{(0,1)}[(A + \gamma I)^{-1}, (B + \mu I)^{-1}] &= \left[\left(\frac{1/\gamma}{1/\mu} \right)^r - 1 \right] \left(1 - r \log \frac{1/\gamma}{1/\mu} \right) \\ &\quad - \left(\frac{1/\gamma}{1/\mu} \right)^r \log \det_{2X}([(B + \mu I)(A + \gamma I)^{-1}]^r) \\ &= \left[\left(\frac{\mu}{\gamma} \right)^r - 1 \right] \left(1 - r \log \frac{\mu}{\gamma} \right) - \left(\frac{\mu}{\gamma} \right)^r \log \det_{2X}([(A + \gamma I)(B + \mu I)^{-1}]^{-r}) \\ &= \left[\left(\frac{\gamma}{\mu} \right)^{-r} - 1 \right] \left(1 + r \log \frac{\gamma}{\mu} \right) - \left(\frac{\gamma}{\mu} \right)^{-r} \log \det_{2X}([(B + \mu I)^{-1}(A + \gamma I)]^{-r}) \\ &= D_{-r}^{(0,1)}[(A + \gamma I), (B + \mu I)], \end{aligned}$$

where we have used the property $\det_{2X}([(A + \gamma I)(B + \mu I)^{-1}]^{-r}) = \det_{2X}([(B + \mu I)^{-1}(A + \gamma I)]^{-r})$ by Lemma 6. This completes the proof. \square

Proof of Theorem 8 (Affine invariance). For any $(A + \gamma I), (B + \mu I) \in \mathcal{P}\mathcal{C}_2(\mathcal{H})$, and any $(C + \nu I) \in \text{HS}_X(\mathcal{H})$, we have

$$\begin{aligned} & (C + \nu I)(A + \gamma I)(C + \nu I)^* \\ &= CAC^* + \nu(CA + AC^*) + \nu^2 A + \gamma CC^* + \gamma\nu(C + C^*) + \gamma\nu^2 I \in \mathcal{P}\mathcal{C}_2(\mathcal{H}), \\ & (C + \nu I)(B + \mu I)(C + \nu I)^* \\ &= CBC^* + \nu(CB + BC^*) + \nu^2 B + \mu CC^* + \mu\nu(C + C^*) + \mu\nu^2 I \in \mathcal{P}\mathcal{C}_2(\mathcal{H}), \end{aligned}$$

For two operators $(A + \gamma I), (B + \mu I) \in \mathcal{P}\mathcal{C}_2(\mathcal{H})$, we then have

$$\begin{aligned} & [(C + \nu I)(A + \gamma I)(C + \nu I)^*][(C + \nu I)(B + \mu I)(C + \nu I)^*]^{-1} \\ &= (C + \nu I)[(A + \gamma I)(B + \mu I)^{-1}](C + \nu I)^{-1}. \end{aligned}$$

Then for any $p \in \mathbb{R}$, we have by Proposition 1,

$$\begin{aligned} & ([(C + \nu I)(A + \gamma I)(C + \nu I)^*][(C + \nu I)(B + \mu I)(C + \nu I)^*]^{-1})^p \\ &= [(C + \nu I)[(A + \gamma I)(B + \mu I)^{-1}](C + \nu I)^{-1}]^p \\ &= (C + \nu I)[(A + \gamma I)(B + \mu I)^{-1}]^p(C + \nu I)^{-1} \in \text{HS}_X(\mathcal{H}) \end{aligned}$$

Thus for the cases $\alpha = 0, \beta > 0$ and $\alpha > 0, \beta = 0$, the affine-invariance follows from the Similarity Invariance of the extended Hilbert-Carleman determinant \det_{2X} , stated in Lemma 4, along with the invariance of the ratio $\frac{\gamma\nu^2}{\mu\nu^2} = \frac{\gamma}{\mu}$.

For the case $\alpha > 0, \beta > 0$, let $a = \frac{\alpha}{\alpha+\beta}, b = \frac{\beta}{\alpha+\beta}, p = r(1 - \delta), q = r\delta$, we have by Proposition 3

$$a[(A + \gamma I)(B + \mu I)^{-1}]^p + b[(A + \gamma I)(B + \mu I)^{-1}]^{-q} \in \text{Tr}_X(\mathcal{H}).$$

It follows then that

$$\begin{aligned} & a([(C + \nu I)(A + \gamma I)(C + \nu I)^*][(C + \nu I)(B + \mu I)(C + \nu I)^*]^{-1})^p \\ &+ b([(C + \nu I)(A + \gamma I)(C + \nu I)^*][(C + \nu I)(B + \mu I)(C + \nu I)^*]^{-1})^{-q} \\ &= (C + \nu I)(a[(A + \gamma I)(B + \mu I)^{-1}]^p + b[(A + \gamma I)(B + \mu I)^{-1}]^{-q})(C + \nu I)^{-1} \\ &\in \text{Tr}_X(\mathcal{H}). \end{aligned}$$

From the Similarity Invariance of both the extended Fredholm determinant \det_X , stated in Eq. (34), along with the invariance of the ratio $\frac{\gamma\nu^2}{\mu\nu^2} = \frac{\gamma}{\nu}$, we obtain the affine-invariance for $D_r^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)]$. \square

Proof of Theorem 9 (Unitary invariance). The proof for this theorem is similar to that of Theorem 9, by utilizing the fact that $C^* = C^{-1}$ and the Similarity Invariance

$$\det_X[C(A + \gamma I)C^{-1}] = \det_X(A + \gamma I), \quad A + \gamma I \in \text{Tr}_X(\mathcal{H}),$$

for the case $\alpha > 0, \beta > 0$, and

$$\det_{2X}[C(A + \gamma I)C^{-1}] = \det_{2X}(A + \gamma I), \quad A + \gamma I \in \text{HS}_X(\mathcal{H}),$$

for the cases $\alpha > 0, \beta = 0$ and $\alpha = 0, \beta > 0$. \square

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